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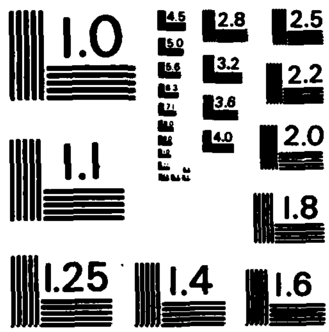

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AN INEQUALITY CONCERNING THE DEVIATION BETWEEN THEORETICAL AND EMPIRICAL DISTRIBUTIONS*

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1. INTRODUCTION

The result. Let $x_1,\ldots,x_r$ be r points in R^d, and $\mathcal{A}$ be a class of Borel sets in R^d. Denote by $\Delta^{\mathcal{A}}(x_1,\ldots,x_r)$ the number of distinct sets in $\{\{x_1,\ldots,x_r\} \cap A,\ A \in \mathcal{A}\}$. Define

$$m^{\mathcal{A}}(r) = \max_{x_1,\ldots,x_r \in R^d} \Delta^{\mathcal{A}}(x_1,\ldots,x_r).$$

Vapnik and Chervonenkis (1971) showed that either $m^{\mathcal{A}}(r) = 2^r$ for any positive integer r or $m^{\mathcal{A}}(r) \leq r^s+1$, where s is the smallest k such that $m^{\mathcal{A}}(k) \neq 2^k$. A class of sets $\mathcal{A}$ for which the latter case holds will be called a V-C class with index s.

Suppose that μ is a probability measure on R^d. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random vectors with common distribution μ, and μ_n be the empirical distribution of $X_1,\ldots,X_n$. Denote a "distance" between μ_n and μ by

$$D_n(\mathcal{A},\mu) = \sup_{A\in\mathcal{A}} |\mu_n(A) - \mu(A)|.$$

Throughout this paper we assume that $D_n(\mathcal{A},\mu)$, $\sup_{A\in\mathcal{A}}|\mu_n(A)-\mu_{2n}(A)|$ and $\sup_{A\in\mathcal{A}} \mu_n(A)$ are all random variables. We shall prove the following

<u>Theorem 1</u>. Let $\mathcal{A}$ be a V-C class with index s such that

$$\sup_{A\in\mathcal{A}} \mu(A) \leq \delta \leq 1/8. \tag{1}$$

Then for any $\varepsilon > 0$ we have

$$P\{D_n(\mathcal{A},\mu)>\varepsilon\} \leq 5(2n)^s \exp(-n\varepsilon^2/(91\delta+4\varepsilon)) \tag{2}$$

$$+ 7(2n)^s \exp(-\delta n/68)$$

$$+ 2^{2+s} n^{1+2s} \exp(-\delta n/8),$$

provided $n \geq \max\left(12\sigma/\varepsilon^2,\ 68(1+s)(\log 2)/\delta\right)$.

The proof of (2) is based on an important inequality proved by Devroye and Wagner (1980).

2. HISTORICAL NOTES

A few remarks concerning this inequality are in order. In 1971, Vapnik and Chervonenkis proved that, for any $\varepsilon > 0$

$$P\{D_n(A,\mu) > \varepsilon\} \leq 4\exp(-n\varepsilon^2/8)\ E\Delta^A(X_1,\ldots,X_{2n}). \tag{3}$$

This inequality is quite general since no restrictions such as (1) are imposed. In using this inequality, an extimate of $m^A(n)$ must be given, see, for example, Gaenssler and Stute (1979), Wenocur and Dudley (1981).

The weakness of (3) lies in the fact that, in many applications $\varepsilon = \varepsilon_n \to 0$ as $n\to\infty$. In this case $n\varepsilon_n^2$ may not tend to ∞ or tend to ∞ very slowly. For this reason, the inequality proved by Devroye and Wagner (1980) is sometimes more useful. They proved that, if $\sup_A \mu(A) \leq \delta \leq \frac{1}{4}$, then for any $\varepsilon > 0$

$$\begin{aligned} P\{D_n(A,\mu) > \varepsilon\} &\leq 4m^A(2n)\exp(-n\varepsilon^2/(64\delta+4\varepsilon)) \\ &+ 2P\{\operatorname{Sup}_A \mu_{2n}(A) > 2\delta\} \end{aligned} \tag{4}$$

for $n \geq 8\delta/\varepsilon^2$. If we further have

$$\operatorname*{Sup}_{A\varepsilon A}\ \operatorname*{Sup}_{x,y\varepsilon A} ||x-y|| \leq \rho < \infty$$

and

$$\operatorname*{Sup}_{x\varepsilon R^d} \mu(S(x,\rho)) \leq \delta \leq \tfrac{1}{4}, \tag{5}$$

here $||\cdot||$ is the L_2 or L_∞ norm in R^d, and $S(x,\rho)$ is the closed ball with radius ρ

centered at x, then

$$P\{D_n(A,\mu)>\varepsilon\} \leq 4m^A(2n)\exp(-n\varepsilon^2/(64\delta+4\varepsilon)) + 4n\exp(-n\delta/10) \tag{6}$$

for $n \geq \max(1/\delta, 8\delta/\varepsilon^2)$.

This inequality is most useful when A is the class of balls with the same diameter (norm L_2 or L_∞). Otherwise δ may be much larger than $\mathrm{Sup}_A\, \mu(A)$, and (6) gives no improvement over (3). Chen and Zhao (1984) made an essential improvement in the one-dimensional case:

Let A be a class of intervals in R^1, satisfying $\sup_{I\in A} \mu(I) \leq \delta \leq 1$.

Then there exists positive absolute constants $C_0, C_1, \ldots, C_4$ such that for any $\varepsilon > 0$

$$P\{\sup_{I\in A}|\mu_n(I) - \mu(I)| > \varepsilon\} \leq C_1\varepsilon^{-1}\sqrt{\delta/n}\exp(-C_2 n\varepsilon^2/\delta) + C_3\exp(-C_4 n\varepsilon), \tag{7}$$

provided $n/\log n > C_0/\varepsilon$.

The proof of (7) relies on a result concerning the strong approximation to Brownian bridge of the empirical process on R^1. The argument fails in the general case $d > 1$. The inequality (2), to be proved in the next section, gives a satisfactory generalization to the case $d \geq 1$.

3. PROOF OF THEOREM 1

Set

$$\delta_j = 2^{2^{-1}+2^{-2}+\dots+2^{-j}}\delta, \quad j=1,2,\dots,r,$$

where r will be chosen later. Then

$$\delta < \delta_1 < \delta_2 < \dots < \delta_r < 2\delta \leq \tfrac{1}{4}$$

When $n \geq 12\delta/\varepsilon^2$ we have $n \geq 8\delta_1/\varepsilon^2$. From (4), the definition of V-C class and the fact that

$$\mathrm{Sup}_A \ \mu(A) \leq \delta_1 \leq \tfrac{1}{4},$$

it follows that

$$P\{D_n(A,\mu)>\varepsilon\} \leq 4\{(2n)^s+1\} \exp(-n\varepsilon^2/(64\delta_1+4\varepsilon))$$

$$+ 2P\{\mathrm{Sup}_A \ \mu_{2n}(A) > 2\delta_1\}$$

$$\leq 5(2n)^s\exp(-n\varepsilon^2/(64\sqrt{2}\delta+4\varepsilon)) + 2P\{D_{2n}(A,\mu) > \delta_1\},$$

provided $n \geq 12\delta/\varepsilon^2$.

When $\delta n \geq 68(1+s)\log 2$, we have $2^{j-1}n \geq 8\delta_j/\delta_{j-1}^2$ for $j = 2,3,\dots r$. As before, from (4) and $\mathrm{Sup}_A \ \mu(A) \leq \delta_2 \leq \frac{1}{4}$, it follows that

$$P\{D_n(A,\mu) > \varepsilon\} \leq 5(2n)^s\exp(-n\varepsilon^2/(91\delta+4\varepsilon))$$

$$+(2\cdot 5)(2\cdot 2n)^s\exp(-2n\delta_1^2/(64\delta_2+4\delta_1))$$

$$+ \ 2^2 \ P\{D_{2^2 n}(A,\mu) > \delta_2\},$$

provided $n \geq \max(68(1+s)\log 2/\delta,\ 12\delta/\varepsilon^2)$.

Using (4) and $\sup_A \mu(A) \leq \delta_j \leq \frac{1}{4}$ repeatedly, we obtain

$$P\{D_n(A,\mu) > \varepsilon\} \leq 5(2n)^s \exp(-n\varepsilon^2/(91\delta+4\varepsilon))$$
$$+ \sum_{j=1}^{r-1} 2^j \cdot 5(2^j \cdot 2n)^s \exp(-2^j n\delta_j^2/(68\delta_{j+1})) \tag{8}$$
$$+ 2^r P\{D_{2^r n}(A,\mu) > \delta_r\} \triangleq J_{1,n} + J_{2,n} + J_{3,n},$$

provided $n \geq \max(68(1+s)\log 2/\delta,\ 12\delta/\varepsilon^2)$.

It is easy to see that

$$2^j \delta_j^2/\delta_{j+1} \geq 2j\delta, \quad j=1,\ldots, r-1. \tag{9}$$

Hence it follows from (8), (9) and $2^{1+s} \leq e^{\delta n/68}$ that

$$J_{2,n} \leq 5(2n)^s \sum_{j=1}^{r-1} 2^{(1+s)j} \cdot \exp(-2^j n\delta_j^2/(68\delta_{j+1}))$$
$$\leq 5(2n)^s \sum_{j=1}^{\infty} (2^{1+s})^j \exp(-2j\delta n/68)$$
$$\leq 5(2n)^s \sum_{j=1}^{\infty} \exp(-j\delta n/68)$$
$$= 5(2n)^s e^{-\delta n/68}(1-e^{-\delta n/68})^{-1}$$
$$\leq 5(2n)^s(1-2^{-(1+s)})^{-1} e^{-\delta n/68}$$
$$\leq 7(2n)^s \exp(-\delta n/68),$$

where $s \geq 1$ is invoked.

When $\delta n \geq 68(1+s)\log 2$, we have $2^r n\delta_r \geq 2$. By (3)

$$J_{3,n} \leq 2^{r+1}((2^{r+1}n)^s+1)\exp(-2^r n\delta_r^2/8). \tag{11}$$

Take $r = r_n$ to be an integer such that $n/2 < 2^r \leq n$. When $\delta n \geq 68(1+s)\log 2$, we have $n^2\delta_r^2 \geq 2$, $n\delta_r \geq \sqrt{2}$ and $n\delta_r^2 \geq 2\delta$. By (11) we have

$$\begin{aligned} J_{3,n} &\leq 2n((2n^2)^s+1)\exp(-n^2\delta_r^2/16) \\ &\leq 4n(2n^2)^s\exp(-\delta n/8). \end{aligned} \tag{12}$$

Formula (2) follows from (8), (10) and (12). The theorem is proved.

4. APPLICATIONS

Theorem 1 has some applications in strong convergence problems involving the uniform deviation between frequencies and probabilities of a class of events. As an example, we consider the nearest neighbor (NN) density estimates proposed by Loftsgarden and Quesenberry (1965). Suppose that X is a R^d-valued random vectors with distribution μ and unknown density function f. The so called NN estimate of f(x) has the form

$$\hat{f}_n(x) = {}^k/\{n(2a_n(x))^d\},\ x = (x^{(1)},\ldots,x^{(d)})\varepsilon R^d, \tag{13}$$

where $k = k_n \leq n$ is a positive integer chosen in advance, $a_n(x)$ is the smallest $a > 0$ such that the cube $[x-a,x+a] = \Pi_{i=1}^d [x^{(i)}-a,x^{(i)}+a]$ contains at least k sample points. As an application of Theorem 1, we prove a theorem about the convergence rate of $\mathrm{Sup}_{x\varepsilon R^d} |\hat{f}_n(x) - \hat{f}(x)|$

In the sequel, we use c, α, c_1, c_2, ... for some positive constants independent of n and x. For $x = (x^{(1)},\ldots,x^{(d)})\varepsilon R^d$, $y = (y^{(1)},\ldots,y^{(d)})\varepsilon R^d$, write $f'(x)(y-x) = \sum_{i=1}^d \frac{\partial f}{\partial x^{(i)}} (y^{(i)}-x^{(i)})$, and take $||y - x|| = \max_{1\leq i\leq d}|y^{(i)} - x^{(i)}|$. We say that the density function f belongs to λ-class for some $\lambda\varepsilon(0,2]$, if $\lambda\varepsilon(0,1]$ and $|f(y) - f(x)| \leq C||y-x||^\lambda$ for any $x,y\varepsilon R^d$, or $\lambda\varepsilon(1,2]$ and, f are bounded and

$$|f(y) - f(x) - f'(x)(y-x)| \leq C||y - x||^\lambda$$

for any $x,y\varepsilon R^d$. We have

<u>Theorem 2</u>. Suppose that f belongs to λ-class for some $\lambda\varepsilon(0,2]$. Take $k = o(n)$ and

$$^k/n \geq \beta \left(\frac{\log n}{n}\right)^{(d+\lambda)/(d+3\lambda)} \tag{14}$$

where $\beta > 0$ is any given constant. Then

$$\lim_{n\to\infty} \sup\{(^n/k)^{\lambda/(d+\lambda)} \operatorname{Sup}_x |\hat{f}_n(x) - \dot{f}(x)|\} \leq C \text{ a.s.} \tag{15}$$

To prove this theorem, we need the following lemma. In the sequel, μ_n denotes the empirical measure of $X_1,\ldots,X_n$. Besides, a cube of the form [x-a,x+a] is called a regular cube.

<u>Lemma 3.</u> Let A be a class of regular cubes satisfying the measurability conditions mentioned in paragraph 1 and the condition

$$\operatorname{Sup}_{A\epsilon A} \mu(A) \leq k/n \leq 1/8.$$

Take $k = o(n)$ and

$$k/n \geq \beta \left(\frac{\log n}{n}\right)^{1/(1+2r)}, \tag{16}$$

where $r > 0$ and $\beta > 0$ is any given constant. Then

$$\lim_{n\to\infty} \sup\{(\tfrac{n}{k})^{1+r} \sup_{A\epsilon A} |\mu_n(A) - \mu(A)|\} \leq C_1 \quad \text{a.s.}$$

Notice that A is a V-C class, one can obtain Lemma 3 from Theorem 1 immediately. The proof is omitted.

Proof of Theorem 2. Take $k = o(n)$ and

$$k/n \geq \beta(^{\log n}/n)^{(d+\lambda)/(d+3\lambda)}$$

Put

$$V_n = \theta_1^{-1}(^k/n)^{\lambda/(d+\lambda)}$$

$$q_n = \theta_2 V_n = \theta_1^{-1}\theta_2(^k/n)^{\lambda/(d+\lambda)}$$

$$B_n = \{x: f(x) \geq V_n\}$$

where θ_1, $\theta_2 \varepsilon (0,1)$ will be chosen later.

Let $\mu(x,a)$ and $\mu_n(x,a)$ be the probability measure and empirical measure of $[x-a,x+a]$ respectively. Put $M = \max(\operatorname{Sup}_x f(x),1)$. We have

$$P\{\operatorname{Sup}_{x\varepsilon B_n} |\hat{f}_n(x) - f(x)| > q_n\} \leq I_n + J_n \tag{17}$$

where

$$I_n = P(U_{x\varepsilon B_n}\{\hat{f}_n(x) > f(x) + q_n\}),$$
$$J_n = P(U_{x\varepsilon B_n}\{\hat{f}_n(x) < f(x) - q_n\}). \tag{18}$$

Thus

$$I_n \leq P(U_{x\varepsilon B_n}\{a_n(x) < b_n(x)\}), \tag{19}$$

where

$$2b_n(x) = \{\frac{k}{nf(x)}\,(1+q_n/f(x))^{-1}\}^{1/d}.$$

Fix $x\varepsilon B_n = \{x: f(x) \geq V_n\}$. Take $\theta_2 < 1/8$, then $q_n/f(x) \leq \theta_2 < 1/8$. Noticing $^1/(1+t) < 1 - 7t/8$ for $0 \leq t < 1/8$, we have

$$2b_n(x) \leq \{\frac{k}{nf(x)}\,(1-7q_n/8f(x))\}^{1/d}$$
$$\leq (k/nf(x))^{1/d}.$$

It follows that

$$\mu(x,b_n(x)) = \int_{x-b_n(x)}^{x+b_n(x)} f(t)dt$$

$$\leq (2b_n(x))^d f(x) + C_2(2b_n(x))^{d+\lambda}$$

$$= (2b_n(x))^d f(x)[1 + C_2(2b_n(x))^\lambda/f(x)]$$

$$\leq \frac{k}{n}(1-\frac{7}{8}q_n/f(x))(1+C_2(\frac{k}{nf(x)})^{\lambda/d}/f(x))$$

$$\leq \frac{k}{n}(1-\frac{7}{8}q_n/f(x) + C_2(\frac{k}{nf(x)})^{\lambda/d}/f(x)).$$

Fix θ_2, take θ_1 small enough such that $C_2\theta_1^{(\lambda+d)/d} < \frac{3}{8}\theta_2$, then $C_2(\frac{k}{nf(x)})^{\lambda/d}$ $\leq C_2\theta_1^{\lambda/d}(k/n)^{\lambda/(\lambda+d)} < \frac{3}{8}\theta_1^{-1}\theta_2(k/n)^{\lambda/(d+\lambda)} = \frac{3}{8}q_n$. It follows that

$$\mu(x,b_n(x)) \leq \frac{k}{n}(1-\frac{1}{2}q_n/f(x)) < k/n,$$

and

$$\frac{k}{n} - \mu(x,b_n(x)) \geq kq_n/(2nM).$$

Hence, by (19) and Theorem 1, we have

$$I_n \leq P\{\sup_{x\in B_n}(\mu_n(x,b_n(x)) - \mu(x,b_n(x)) \geq kq_n/(2nM)\}$$

$$\leq C_5 n^\alpha\{\exp(-\frac{n(kq_n/2nM)^2}{91k/n+2kq_n/nM}) + \exp(-k/68)\}$$

where α is a constant depending only on d. In view of (14), we have for large n

$$I_n \leq C_5 n^\alpha\{\exp(-\theta_1^{-1}\theta_2^2 M^{-2}\beta^{1+2\lambda/(d+\lambda)}\log n/400)$$

$$+ \exp(-k/68)\}.$$

Take θ_1 small enough, we have

$$\sum I_n < \infty. \tag{20}$$

In the same way, we can take θ_1 and θ_2 such that

$$\sum J_n < \infty \tag{21}$$

By (17), (18), (20) and (21), we have

$$\sum P\{q_n^{-1} \sup_{x \in B_n} |\hat{f}_n(x) - f(x)| > 1\} < \infty.$$

By Borel-Cantelli's lemma,

$$\lim_{n\to\infty} \sup\{q_n^{-1} \sup_{x \in B_n} |\hat{f}_n(x) - f(x)|\} \leq 1 \text{ a.s.} \tag{22}$$

Fix θ_1, θ_2, and take $2b_n = C_3(k/n)^{1/(d+\lambda)}$. Fix $x \varepsilon B_n^c = \{x: f(x) < V_n\}$. With small C_3 we have

$$\begin{aligned}
\mu(x,b_n) &= \int_{x-b_n}^{x+b_n} f(t)dt \\
&\leq (2b_n)^d f(x) + C_2(2b_n)^{d+\lambda} \\
&\leq \frac{k}{n}[\theta_1^{-1} C_3^d + C_2 C_3{}^{d+\lambda}] < k/2n < k/n.
\end{aligned}$$

Taking $r = \lambda/(d+\lambda)$ in Lemma 3, we can assert with probability one that, for n large enough, the inequality

$$\begin{aligned}
\mu_n(x,b_n) &\leq \mu(x,b_n) + 2C_1(k/n)^{(d+2\lambda)/(d+\lambda)} \\
&< k/2n + 2C_1(k/n)^{(d+2\lambda)/(d+\lambda)} < k/n
\end{aligned}$$

holds uniformly for $x \varepsilon B_n^c$. By definition, for $x \varepsilon B_n^c$,

$$a_n(x) \geq b_n = \frac{1}{2} C_3(k/n)^{1/(d+\lambda)},$$

$$\hat{f}_n(x) \geq C_4(k/n)^{\lambda/(d+\lambda)}$$

It follows that

$$\lim_{n\to\infty} \text{Sup}\{(n/k)^{\lambda/(d+\lambda)} \sup_{x\varepsilon B_n^c} |\hat{f}_n(x) - f(x)|\} \leq C_4 \text{ a.s.} \tag{23}$$

Theorem 2 is proved in view of (22) and (23).

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In this paper the author established an inequality concerning the uniform deviation between theoretical and empirical distributions. An application in strong convergence of nearest neighbor density estimate is also discussed.

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